

Hedging When The Money Runs Out

Review of Yumiharu Nakano, *Optimal Hedging in the Presence of Shortfall Risk* (doctoral thesis, Hokkaido University technical report series in mathematics 94, 2005, 45 pp.)

TLDR / 80-20

Underfunded, the optimal hedge picks scenarios — it does not shrink the position.

Solve for a $[0,1]$ test saying which states to defend, then replicate perfectly on those. Budget sets the cutoff.

The Keys

1. The problem, and the switch that makes it new.

Perfect hedging costs x_0 ; start below it and shortfall is certain, so the question is which strategy minimises the damage. Föllmer–Leukert measured damage as expected loss. Nakano swaps in **coherent risk measures**, and his first theorem proves every one of them equals *the worst expected loss across a set of scenarios*.

2. The answer's shape: hedge $\phi \cdot H$ — shrunk, not diluted.

The optimal strategy is to perfectly hedge a modified claim ϕH , where $\phi \in [0,1]$ is a "randomized test". In a complete market the Neyman–Pearson lemma makes it explicit: defend wherever a density ratio exceeds a threshold set by your capital. Black–Scholes with a European call: **hedge only when the stock finishes above a cutoff**, set by your budget.

3. The worked examples do not exhibit the behaviour the theory predicts.

The one-period binomial with short-selling banned collapses to $\hat{X} = (x/x_0)H$ — plain proportional scaling, because with a single paying state there is nothing to select between. That is the opposite of the headline claim, and it is the only constrained example given.

4. Part III buys a closed form by dropping a safety constraint.

In the jump-diffusion market he keeps only an integrability condition on wealth, not non-negativity. That splits the problem cleanly — hedge H perfectly, separately minimise a power loss — so the optimal portfolio is $\pi_H - \pi_0$, in closed form. The price is a strategy that may run the account negative on the way.

AI's Take

Three of the author's own published papers bound together: every theorem cites back to [21], [22] or [23], so the thesis contributes the stitching, not the results. Proofs fill most of the 45 pages, nothing meets a market, and a strategy that dips negative is not implementable anywhere that margin-calls.

The idea worth taking: **partial protection is a selection problem, not a scaling problem**. Rank scenarios by protection bought per unit of cost, buy full cover down the list until the money runs out, and leave the rest uncovered on purpose. That is an 80/20 priority queue with a budget constraint.

One line to keep: the moment you define risk as "worst case over a set of scenarios", every optimal-hedging question becomes a hypothesis test — and your budget is the significance level.