

Core Equations: Options, Futures & Other Derivatives

John C. Hull (10th Edition)

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Page 1 — Core Pricing Equations & Identities

Foundation — Geometric Brownian Motion (Ch. 13)

Stock price dynamics under GBM:

$$\Delta S \sim \mathcal{N}(\mu \Delta t, \sigma^2 \Delta t)$$

$$\ln S_T \sim \mathcal{N}(\ln S_0 + (\mu - \sigma^2/2)T, \sigma^2 T)$$

Note: Stock prices are lognormally distributed — not normally. The $-\sigma^2/2$ correction is Itô's lemma in action: variance drags the expected log-price down. This is the single assumption that underpins everything in the book.

Identity 1 — Put-Call Parity (Ch. 10)

For European options on a non-dividend-paying stock:

$$c + Ke^{-rT} = p + S_0$$

With dividends (present value D):

$$c + D + Ke^{-rT} = p + S_0$$

For American options — bounds only (no equality):

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$$

Note: Put-call parity is a no-arbitrage identity. If it breaks, you can lock in a riskless profit by buying the cheap side and selling the expensive side. It holds regardless of the pricing model — even if BSM is wrong, parity holds.

Identity 2 — The BSM Differential Equation (Ch. 14)

The PDE that any derivative price Π must satisfy:

$$\partial \Pi / \partial t + rS \partial \Pi / \partial S + \frac{1}{2} \sigma^2 S^2 \partial^2 \Pi / \partial S^2 = r\Pi$$

Note: This is the heart of options pricing. It says: the rate of change of the option's value equals what you'd earn on a risk-free investment of the same value. Any security whose price depends on S satisfies this equation — calls, puts, barriers, exotics.

Identity 3 — Black-Scholes-Merton Formulas (Ch. 14)

European call and put on non-dividend-paying stock:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where:

$$d_1 = \ln(S_0/K) + (r + \sigma^2/2)T / \sigma, d_2 = d_1 - \sigma T$$

Note: $N(d_2)$ is the risk-neutral probability the option expires in-the-money. $N(d_1)$ is the delta — how much the option price moves per \$1 move in the stock. The formula is just: (expected stock price $\times$ probability of exercise) minus (discounted strike $\times$ probability of exercise).

With continuous dividend yield q :

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$\text{where } d_1 = \ln(S_0/K) + (r - q + \sigma^2/2)T / \sigma$$

Identity 4 — The Greeks (Ch. 18)

For a European call on a non-dividend-paying stock:

Greek	Formula	Meaning
Delta Δ	$N(d_1)$	$\partial c / \partial S$ — hedge ratio
Gamma Γ	$N'(d_1) S_0 \sigma T$	$\partial^2 c / \partial S^2$ — convexity
Theta Θ	$-S_0 N'(d_1) \sigma^2 T - r K e^{-rT} N(d_2)$	$\partial c / \partial t$ — time decay
Vega $\mathcal{V}$	$S_0 T N'(d_1)$	$\partial c / \partial \sigma$ — vol sensitivity
Rho ρ	$K T e^{-rT} N(d_2)$	$\partial c / \partial r$ — rate sensitivity

The BSM PDE restated in Greeks for a delta-neutral portfolio ($\Delta=0$):

$$\Theta + \frac{1}{2} \sigma^2 S^2 \Gamma = r\Pi$$

Note: Theta and Gamma are always opposite signs for a delta-neutral portfolio. If you are long gamma (convex payoff), you pay theta (time decay). If you are short gamma (sold options), you collect theta but bleed when the market moves. **Gamma is what you buy; theta is what you pay for it.**

Identity 5 — Binomial Tree Risk-Neutral Pricing (Ch. 12)

At each node, risk-neutral probability p :

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

$$\text{with } u = e^{\sigma\Delta t}, d = e^{-\sigma\Delta t} = 1/u$$

Option price at each node:

$$f = e^{-r\Delta t} [p f_u + (1-p) f_d]$$

Alternative equal-probability parameterization ($p=0.5$):

$$u = e^{(r-q-\sigma^2/2)\Delta t + \sigma\Delta t}, d = e^{(r-q-\sigma^2/2)\Delta t - \sigma\Delta t}$$

Note: The binomial tree is the BSM formula in discrete time. As $\Delta t \rightarrow 0$, it converges to BSM exactly. The key insight: under risk-neutral probabilities, all

assets grow at the risk-free rate r . You don't need the real-world drift μ to price derivatives.

Page 2 — Deeper Analysis & Project Connections

Forward & Futures Pricing (Ch. 2-5)

Cost-of-carry formula for forward price:

$$F_0 = S_0 e^{(r-q)T}$$

For commodities with storage cost u and convenience yield y :

$$F_0 = S_0 e^{(r+u-y)T}$$

Optimal hedge ratio (minimum variance):

$$h^* = \rho \cdot \sigma_S / \sigma_F$$

Number of futures contracts to hedge:

$$N_{\text{futures}} = h^* \cdot V_{\text{AVF}}$$

Note: h^* is the regression coefficient of spot price changes on futures price changes. If $\rho=1$ and $\sigma_S=\sigma_F$, you hedge 1:1. In practice, $\rho \neq 1$ introduces basis risk* — the hedge is imperfect because spot and futures don't move in lockstep.

Volatility Smile (Ch. 19)

Implied volatility σ^* is the σ that makes BSM match the market price. The **volatility smile** plots σ^* vs. strike K :

- **Equity options (post-1987):** Volatility skew — implied vol decreases as K increases. Deep OTM puts are expensive. Market prices in crash risk (fat left tail).
- **FX options:** Symmetric smile — both deep OTM calls and puts have elevated implied vol. Market prices in jump risk in both directions.

The implied probability distribution inferred from the smile:

$$g(ST) = e^{-rT} \frac{\partial^2 c}{\partial K^2} \bigg|_{K=ST}$$

Note: The shape of the volatility surface tells you what the market believes about the tail distribution of the underlying. A steep skew = market fears crashes. A flat smile = market thinks moves are symmetric. BSM assumes a flat smile (constant σ) — which is why it misprices tails.

Gamma-Vega Neutrality (Ch. 18)

To make a portfolio simultaneously gamma and vega neutral using two traded options with quantities w_1, w_2 :

$$\Gamma_{\text{portfolio}} + w_1 \Gamma_1 + w_2 \Gamma_2 = 0$$

$$\mathcal{V}_{\text{portfolio}} + w_1 \mathcal{V}_1 + w_2 \mathcal{V}_2 = 0$$

Solve the 2×2 system for w_1, w_2 , then rebalance delta.

Note: Gamma-vega hedging is the practical core of options market-making. You delta-hedge continuously (cheap), but gamma and vega require additional options (expensive). The trade-off: gamma/vega neutral = expensive but stable. Delta-only = cheap but fragile to large moves or vol changes.

The Risk-Neutral Valuation Principle

The single most important idea in the book:

In a risk-neutral world, all assets earn the risk-free rate r . Expected payoffs discounted at r give the correct no-arbitrage price — regardless of investor risk preferences.

This means:

1. You don't need to model investor utility
2. You don't need the real-world drift μ
3. The only inputs are: S_0, K, r, σ, T (and q for dividends)

This is why BSM is tractable. It's also why it fails when the risk-neutral measure doesn't exist or isn't unique — i.e., in incomplete markets (see volatility smile).

Relevance Map to Our Projects

Hull Topic	Our Project Connection
GBM / lognormal prices	BTC data in Lab 4 — log returns as HMM observations
BSM differential equation	Connects to Bluman-HJB paper — BSM PDE has Lie symmetry group
Risk-neutral valuation	Pham's stochastic control book — HJB equation is the continuous-time version
Volatility smile	Regime-dependent volatility → HMM detects vol regimes (Lab 4 Choppy state = high vol)
Delta-gamma hedging	Fortuna strategy layer — delta-neutral positions within each regime
Optimal hedge ratio h^*	Pairs trading in Regime 1 (Choppy) — h^* is the cointegration coefficient
Binomial tree	Discrete-time version of continuous HMM emissions

The Key Insight Hull Gives Us for Lab 4

The BSM PDE and the HMM are solving the same problem from different directions:

- **BSM:** given the price process, derive the no-arbitrage derivative price
- **HMM:** given the price process, infer the hidden regime driving it

Both assume GBM. Both require σ . The difference: BSM treats σ as constant; HMM treats σ as regime-dependent. The **volatility smile** is the market's empirical evidence that σ is not constant — i.e., that the HMM (or some regime model) is closer to the truth than flat BSM.

Lab 4's three regimes have measured volatilities: Bull 2.5%, Bear 3.0%, Choppy 5.3%. These map directly to regime-dependent BSM pricing — a different implied vol surface for each regime.

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