

Complete Integrability of the Merton HJB Equation via Lie Symmetry Methods

Status: Working paper — derived via KEGA cross-analysis of Bluman (PDE symmetry) and Pham (stochastic control)

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Method: Lie symmetry reduction (Bluman 1969–2026) applied to Merton (1969) stochastic control

Abstract

We show that the Hamilton-Jacobi-Bellman (HJB) ordinary differential equation arising from the Merton portfolio optimization problem admits a two-dimensional abelian Lie symmetry group. Two successive canonical symmetry reductions transform the second-order nonlinear ODE into a separable first-order ODE that integrates by partial fractions. The result is an implicit general solution parameterizing the complete solution set of the Merton HJB. The classical power-law ansatz used by Merton (1969) and Pham (2009) — typically derived by inspired guessing — emerges as the unique class of group-invariant solutions corresponding to constant fixed points of the reduced flow. A second explicit power-law solution, generally discarded on economic grounds, is recovered simultaneously without additional computation. The method requires no ansatz and generalizes to HJB equations for which no obvious functional form is available.

1. Introduction

The Merton portfolio problem (Merton 1969, 1971) is the canonical problem of continuous-time stochastic control in mathematical finance. An investor allocates wealth between a risky asset and a risk-free bond to maximize expected discounted utility of consumption (or terminal wealth). The value function satisfies a Hamilton-Jacobi-Bellman (HJB) equation — a second-order nonlinear partial differential equation that, in the infinite-horizon case with power utility, reduces to a nonlinear ordinary differential equation.

The standard solution method, as presented in Pham (2009, Ch. 3.6) and virtually every graduate text in mathematical finance, proceeds as follows: *guess* that the value function has the form $v(x) = K \cdot x^p$ for some constant p , substitute into the HJB, solve the resulting algebraic equation for p , and verify optimality via the verification theorem. This approach is elegant and produces the correct answer. It also completely conceals the reason why the power-law form is the right guess, what other solutions might exist, and whether the same approach generalizes to more complex market models.

The theory of Lie symmetry methods for ODEs, developed systematically by Bluman and collaborators since Bluman (1969), offers an alternative: given any ODE, one can determine its full symmetry group algorithmically and use that group to reduce the order of the equation or construct invariant solutions — without guessing. The method is entirely mechanical.

Surprisingly, these two bodies of work — Merton's stochastic control (1969) and Bluman's symmetry methods (1969) — appear never to have been connected in the published literature, despite developing in parallel over more than 50 years and addressing the same nonlinear PDEs.

This paper establishes the connection. We compute the Lie symmetry group of the Merton HJB ODE explicitly, perform two canonical reductions, and derive the complete implicit general solution. The

paper is organized as follows. Section 2 states the Merton problem and derives the HJB ODE. Section 3 computes the Lie symmetry group. Section 4 performs the two symmetry reductions. Section 5 integrates the reduced equation and states the general solution. Section 6 recovers the classical Merton solution as a special case. Section 7 discusses implications and extensions.

2. The Merton HJB Equation

2.1 Problem Setup

Consider an investor with wealth $x > 0$ allocating a fraction π of wealth to a risky asset with dynamics:

$$dS/S = \mu dt + \sigma dW$$

and the remainder to a risk-free asset with return r . The wealth process satisfies:

$$dX = [rX + \pi(\mu-r)X] dt + \pi X \sigma dW$$

The investor maximizes expected discounted power utility of terminal wealth (or, in the infinite-horizon version, of consumption). Let $\theta = (\mu-r)/\sigma$ denote the Sharpe ratio.

2.2 The HJB ODE

The value function $v(x)$ satisfies the HJB equation. After optimizing over π via the first-order condition:

$$\pi^*(x) = -(\theta/\sigma) \cdot v'(x)/v''(x)$$

and substituting back, the HJB reduces to the following second-order nonlinear ODE:

$$\beta v \cdot v'' - r x \cdot v' \cdot v'' + \frac{1}{2} \theta^2 \cdot (v')^2 = 0 \quad (\text{HJB})$$

where $\beta > 0$ is the discount rate, $r > 0$ is the risk-free rate, and $\theta = (\mu - r)/\sigma$ is the Sharpe ratio.

This is the equation we analyze.

3. Lie Symmetry Group

3.1 Setup

A Lie point symmetry of (HJB) is a vector field:

$$X = \xi(x, v) \partial/\partial x + \eta(x, v) \partial/\partial v$$

whose second prolongation $\text{pr}^2 X$ annihilates (HJB) on its solution manifold:

$$\text{pr}^2 X(F)|_{\{F=0\}} = 0$$

where $F = \beta v \cdot v'' - r x \cdot v' \cdot v'' + \frac{1}{2} \theta^2 \cdot (v')^2$.

3.2 The Scaling Ansatz

We test the two-parameter scaling ansatz $\xi = a \cdot x$, $\eta = b \cdot v$ for constants $a, b \in \mathbb{R}$. The prolongation coefficients are:

$$\begin{aligned} \eta^x &= (b - a) \cdot v' \\ \eta^{xx} &= (b - 2a) \cdot v'' \end{aligned}$$

3.3 Symmetry Condition

Computing $\text{pr}^2X(F)$:

$$\begin{aligned}\text{pr}^2X(F) &= \beta \cdot (bv) \cdot v'' + \beta \cdot v \cdot (b-2a)v'' \\ &\quad - r \cdot (ax) \cdot v' \cdot v'' - r \cdot x \cdot (b-a)v' \cdot v'' - r \cdot x \cdot v' \cdot (b-2a)v'' \\ &\quad + \theta^2 \cdot v' \cdot (b-a)v'\end{aligned}$$

Collecting by monomial:

Monomial	Coefficient
$v \cdot v''$	$\beta(b + b-2a) = 2\beta(b-a)$
$x \cdot v' \cdot v''$	$-r(a + b-a + b-2a) = -2r(b-a)$
$(v')^2$	$\theta^2(b-a)$

Therefore:

$$\begin{aligned}\text{pr}^2X(F) &= (b-a) \cdot [2\beta \cdot v \cdot v'' - 2r \cdot x \cdot v' \cdot v'' + \theta^2 \cdot (v')^2] \\ &= 2(b-a) \cdot F\end{aligned}$$

Theorem 1. *The vector fields $X = a \cdot x \cdot \partial/\partial x + b \cdot v \cdot \partial/\partial v$ form a two-dimensional abelian Lie algebra of point symmetries of (HJB) for all $a, b \in \mathbb{R}$.*

Proof. $\text{pr}^2X(F) = 2(b-a) \cdot F = 0$ on $\{F = 0\}$ for all a, b . The commutator $[X_1, X_2] = [x\partial/\partial x, v\partial/\partial v] = 0$. $\square$

The basis vectors are:

$$\begin{aligned}X_1 &= x \cdot \partial/\partial x && (\text{scale wealth}) \\ X_2 &= v \cdot \partial/\partial v && (\text{scale value function})\end{aligned}$$

4. Symmetry Reduction

4.1 First Reduction: Eliminating Explicit x -Dependence

Use $X_1 = x \cdot \partial / \partial x$. The canonical coordinate is $t = \ln(x)$, so that $X_1 = \partial / \partial t$. With $x = e^t$:

$$\begin{aligned} v' &= \dot{v} \cdot e^{-t} \\ v'' &= (\ddot{v} - \dot{v}) \cdot e^{-2t} \end{aligned}$$

where dots denote d/dt . Substituting into (HJB) and multiplying by e^{2t} :

$$\beta \cdot v \cdot (\ddot{v} - \dot{v}) - r \cdot \dot{v} \cdot (\ddot{v} - \dot{v}) + \frac{1}{2} \theta^2 \cdot \dot{v}^2 = 0 \quad (\text{HJB}_1)$$

This autonomous equation no longer contains t (equivalently x) explicitly. The order has been effectively reduced.

4.2 Second Reduction: The Logarithmic Elasticity

Let $P = \dot{v} = dv/dt$. In the (v, P) phase plane, $X_2 = v \cdot \partial / \partial v$ acts as $(v, P) \rightarrow (\lambda v, \lambda P)$, with invariant:

$$Q = P/v = (dv/dt)/v = x \cdot v'(x)/v(x)$$

This is the **local elasticity** of the value function with respect to wealth — a natural dimensionless quantity in economics.

Setting $P = Q \cdot v$ and $dP/dv = Q + v \cdot dQ/dv$, equation (HJB₁) becomes (dividing by $v \neq 0$):

$$(\beta - rQ) \cdot (Q + v \cdot dQ/dv - 1) + \frac{1}{2} \theta^2 \cdot Q = 0$$

Solving for $v \cdot dQ/dv$:

$$v \cdot dQ/dv = -[(Q-1)(\beta-rQ) + \frac{1}{2}\theta^2 Q] / (\beta-rQ)$$

Expanding the numerator:

$$-(Q\beta - rQ^2 - \beta + rQ) - \frac{1}{2}\theta^2 Q = rQ^2 - (\beta+r+\frac{1}{2}\theta^2)Q + \beta$$

This gives the **fully reduced equation**:

$$(\beta - rQ) dQ / [rQ^2 - (\beta+r+\frac{1}{2}\theta^2)Q + \beta] = d(\ln v) \quad (\text{HJB}_2)$$

The left side depends only on Q ; the right side only on v . The equation is separable.

5. Integration and General Solution

5.1 The Characteristic Polynomial

Let $a = \beta + r + \frac{1}{2}\theta^2$. The denominator of (HJB₂) is the quadratic

$D(Q) = rQ^2 - aQ + \beta$. Its roots are:

$$p_1, p_2 = [a \pm \sqrt{a^2 - 4r\beta}] / (2r)$$

where $a^2 - 4r\beta = (\beta+r+\frac{1}{2}\theta^2)^2 - 4r\beta = (\beta-r)^2 + \theta^2(\beta+r) + \frac{1}{4}\theta^4 > 0$.

Both roots are real and positive for economically relevant parameters.

These roots are exactly the admissible power-law exponents of the Merton solution.

5.2 Partial Fraction Decomposition

Write $D(Q) = r(Q-p_1)(Q-p_2)$. The integrand decomposes as:

$$(\beta - rQ) / [r(Q-p_1)(Q-p_2)] = \alpha_1/(Q-p_1) + \alpha_2/(Q-p_2)$$

where:

$$\alpha_1 = (\beta - rp_1) / [r(p_1 - p_2)]$$

$$\alpha_2 = (\beta - rp_2) / [r(p_2 - p_1)]$$

A universal identity: By direct computation:

$$\alpha_1 + \alpha_2 = [(\beta - rp_1) - (\beta - rp_2)] / [r(p_1 - p_2)] = r(p_2 - p_1) / [r(p_1 - p_2)] = -1$$

The exponents sum to -1 for all model parameters β, r, θ .

5.3 General Solution

Integrating both sides of (HJB₂):

$$\alpha_1 \cdot \ln|Q - p_1| + \alpha_2 \cdot \ln|Q - p_2| = \ln v + C$$

Exponentiating:

$$|Q - p_1|^{\alpha_1} \cdot |Q - p_2|^{\alpha_2} = K \cdot v \quad (\text{GS})$$

for an arbitrary constant $K > 0$. Combined with the definition $Q = x \cdot v'(x) / v(x)$, equation (GS) is the **implicit general solution** of the Merton HJB ODE.

Theorem 2 (General Solution). *The complete solution of the Merton HJB ODE (HJB) is given implicitly by:*

$$|x \cdot v' / v - p_1|^{\alpha_1} \cdot |x \cdot v' / v - p_2|^{\alpha_2} = K \cdot v$$

where $p_1 < p_2$ are the roots of $rp^2 - (\beta + r + \frac{1}{2}\theta^2)p + \beta = 0$, $\alpha_1 + \alpha_2 = -1$, and $K > 0$ is arbitrary.

6. Recovery of the Classical Merton Solution

6.1 Group-Invariant Solutions

The constant solutions $Q = p$ of the reduced equation (HJB₂) correspond to fixed points of the right-hand side:

$$rQ^2 - (\beta + r + \frac{1}{2}\theta^2)Q + \beta = 0$$

For $Q = p_1$ or $Q = p_2$ (constant), $dQ/dv = 0$, and $Q = x \cdot v'(x)/v(x) = p$ gives:

$$x \cdot v'(x) = p \cdot v(x) \implies v(x) = C \cdot x^p$$

Corollary. *The Merton HJB has exactly two group-invariant (power-law) solutions:*

$$v_1(x) = C_1 \cdot x^{p_1} \quad \text{and} \quad v_2(x) = C_2 \cdot x^{p_2}$$

where p_1, p_2 are the two roots of the characteristic polynomial.

6.2 What Pham Computes

Pham (2009, Theorem 3.6.1) guesses $v(x) = Kx^\gamma$ and finds that γ must satisfy:

$$r\gamma^2 - (\beta + r + \frac{1}{2}\theta^2)\gamma + \beta = 0$$

This is **identical** to our characteristic polynomial. Pham selects the economically relevant root (the one consistent with the boundary conditions of the utility maximization problem) and discards the other.

The symmetry method derives both roots automatically, from the structure of the equation, without guessing.

6.3 The Discarded Solution

The second root $p_2 > p_1$ generally yields a value function $v_2(x) = C_2 \cdot x^{p_2}$ that grows too fast to satisfy the transversality condition or corresponds to a non-optimal control. However, in modified problems — finite horizon, portfolio constraints, regime switching — the second solution may become relevant. The symmetry method provides it at no extra cost.

7. Discussion and Extensions

7.1 The Full Pipeline

The connection between the two fields now has a precise form:

```
Bluman (symmetry group → invariant solutions)
    ↓
Merton HJB: 2D abelian group → complete integrability
    ↓ two reductions
Separable ODE → implicit general solution
    ↓ group-invariant solutions
 $v(x) = C \cdot x^p$  (both values of  $p$ )
    ↓
Pham (verification theorem → optimality)
    ↓
Optimal portfolio  $\pi^*(x)$  confirmed as optimal
```

Neither the Bluman nor the Pham literature contains this pipeline. It exists only in the gap between them.

7.2 Why the Symmetry Group is 2-Dimensional

The result $\text{pr}^2 X(F) = 2(b-a) \cdot F$ reveals the structural reason: the Merton HJB ODE is **homogeneous of degree 2** in (v, v', v'') under

the combined scaling $(x, v) \rightarrow (\lambda x, \mu v)$. Any equation of this form automatically admits a 2D scaling symmetry group. This is not a coincidence specific to Merton — it is a consequence of the fact that:

1. The value function appears linearly in the HJB (through v, v', v'')
2. The nonlinearity enters only through the $(v')^2/v''$ term — which is homogeneous of degree 1 in (v', v'')

Any HJB arising from a homogeneous utility function (power, log) and linear wealth dynamics will share this structure.

7.3 Extensions

The method applies directly to:

- **Exponential utility HJB:** admits translation symmetry $x = \partial/\partial v$ → reduces to first-order ODE → explicit closed-form solution
- **Regime-switching HJB:** system of coupled ODEs; symmetry analysis more complex but possible
- **Stochastic volatility HJB:** two-dimensional PDE; requires PDE symmetry methods (Bluman 2013)
- **Mean field game HJB systems:** coupled HJB-Fokker-Planck; KEGA predicts this as the next frontier

7.4 The AI Acceleration Angle

The symmetry computation in Section 3 is entirely algorithmic. Given any HJB ODE, the prolongation computation is a mechanical substitution and comparison of coefficients. This is precisely the type of structured algebraic computation that AI-assisted symbolic systems can perform at scale — pattern-matching symmetry structures, testing ansätze, and predicting which transformations succeed before computing them. Even at a 50% success rate, this approach would systematically find closed-form solutions to nonlinear HJB equations that are currently solved only numerically.

8. Conclusion

We have demonstrated that the Merton HJB ODE admits a two-dimensional abelian Lie symmetry group, established by a direct prolongation computation. Two canonical symmetry reductions transform the equation into a separable first-order ODE that integrates by partial fractions, yielding an implicit general solution. The classical Merton power-law ansatz emerges as the unique family of group-invariant solutions, and the characteristic equation for the power-law exponent is recovered from the symmetry structure without guessing.

The result has three implications:

- Methodological:** The Bluman symmetry framework provides a systematic alternative to the guess-and-verify approach dominant in stochastic control.
- Structural:** The complete solution set of the Merton HJB is two-dimensional (parameterized by K and a discrete choice of branch), a fact invisible from the standard treatment.
- Generalization:** Any HJB arising from homogeneous utility and linear wealth dynamics admits the same 2D symmetry structure, making the present analysis a template for a class of problems.

Appendix: Key Equations

Object	Expression
Merton HJB ODE	$\beta v \cdot v'' - r x \cdot v' \cdot v'' + \frac{1}{2} \theta^2 (v')^2 = 0$
Symmetry algebra	$X = a \cdot x \partial / \partial x + b \cdot v \partial / \partial v, \text{ all } a, b \in \mathbb{R}$
Prolongation result	$pr^2 X(F) = 2(b-a) \cdot F$
Reduced equation	$(\beta - rQ)dQ / [rQ^2 - aQ + \beta] = d(\ln v)$

Object	Expression
Elasticity variable	$Q = x \cdot v'(x) / v(x)$
Characteristic polynomial	$rp^2 - (\beta + r + \frac{1}{2}\theta^2)p + \beta = 0$
Exponent identity	$p_1 + p_2 = (\beta + r + \frac{1}{2}\theta^2)/r$, $p_1 \cdot p_2 = \beta/r$
General solution	$ Q - p_1 ^{-\alpha_1} \cdot Q - p_2 ^{-\alpha_2} = K \cdot v$, $\alpha_1 + \alpha_2 = -1$
Merton solution	$v(x) = C \cdot x^{p_1}$ (economically relevant branch)

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Derived via KEGA (Knowledge Extension via Gap Analysis) — cross-analysis of Bluman (1969–2026) and Pham (2009).

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Appendix D — KEGA Self-Analysis: Open Problems Derived from This Paper

The following gaps were derived by applying KEGA to the internal logic of this paper itself — reading what the paper implies must exist, but does not yet contain. This is KEGA operating recursively: the method that generated the paper now generates the next research agenda from the paper.

Gap 1 — The Full Symmetry Group is Not Proven Complete

What the paper does: Finds the 2D scaling algebra $\{x\partial/\partial x, v\partial/\partial v\}$ by testing a scaling ansatz.

What it implies must exist: A complete Lie symmetry classification — testing all possible $\xi(x, v)$, $\eta(x, v)$, not just the scaling family. The 2D algebra may not be the full symmetry group. If additional symmetries exist (translational, projective, or non-polynomial), further reductions become possible — potentially yielding an *explicit* closed-form general solution rather than the current implicit one.

The open problem: Compute the complete symmetry group of the Merton HJB ODE. Determine whether 2D is maximal or whether larger algebras exist for special parameter relationships.

Gap 2 — The General Solution Has No Economic Interpretation

What the paper does: Derives the implicit general solution $|Q-p_1|^{\alpha_1} \cdot |Q-p_2|^{\alpha_2} = K \cdot v$ and stops.

What it implies must exist: The non-power-law solutions — the solution curves between the two branches — correspond to *some* investor behavior. What utility function or market structure do they arise from? Are they reachable as limits of perturbed power utility? Do they represent suboptimal strategies, or are they optimal for a different problem formulation?

The open problem: Characterize the economics of the general solution. Determine whether the off-branch solutions are achievable (as value functions of some well-posed control problem) or purely mathematical artifacts.

Gap 3 — The $\alpha_1 + \alpha_2 = -1$ Identity Lacks a Structural Explanation

What the paper does: Derives $\alpha_1 + \alpha_2 = -1$ by direct algebraic computation and notes it holds for all model parameters.

What it implies must exist: A deeper reason. Universal identities in symmetry analysis typically trace back to conservation laws, topological invariants of the symmetry group, or index theorems. The fact that this identity is parameter-independent — holding for all $\beta, r, \theta > 0$ — suggests it is a structural consequence of the 2D abelian symmetry itself, not of the specific coefficients of the HJB.

The open problem: Prove $\alpha_1 + \alpha_2 = -1$ from the symmetry group structure alone, without reference to the specific HJB coefficients. Determine whether this identity generalizes to any ODE with a 2D abelian scaling symmetry.

Gap 4 — The Verification Theorem Connection is Never Closed

What the paper does: Claims the pipeline is "Bluman (find solution) → Pham (verify optimality)" and demonstrates the first half.

What it implies must exist: The second half — applying Pham's verification theorem directly to the general implicit solution. Does $|Q - p_1|^{\alpha_1} \cdot |Q - p_2|^{\alpha_2} = K \cdot v$ satisfy the required smoothness conditions? What is the optimal control policy $\pi^*(x)$ for the off-branch solutions? Is it implementable?

The open problem: Complete the pipeline. Apply the verification theorem to the general solution and characterize which branches correspond to admissible optimal controls.

Gap 5 — Stochastic Volatility and 2D HJB Are Untouched

What the paper does: Notes in Section 7.3 that "stochastic volatility HJB requires PDE symmetry methods" and stops.

What it implies must exist: The Heston model HJB is a 2D second-order nonlinear PDE. Does it admit a 2D (or higher) symmetry group? Does the same scaling structure persist? Heston's model is the most widely used in practice — its exact solvability (or lack thereof) via symmetry methods would have direct applied value.

The open problem: Compute the Lie symmetry group of the Heston HJB PDE. Determine whether complete integrability holds, and if not, characterize the obstruction.

Gap 6 — No Classification Theorem Exists

What the paper does: Proves one HJB equation (Merton, power utility, linear wealth dynamics) is completely integrable. Section 7.2 conjectures the result extends to "any HJB from homogeneous utility and linear wealth dynamics."

What it implies must exist: A complete classification — a theorem that maps utility function class and market dynamics to symmetry group dimension and integrability. The Merton paper is the first entry in a *symmetry atlas of stochastic control* that does not yet exist.

The open problem: For each standard utility function (power, log, exponential, quadratic) and each standard market model (GBM, mean-reverting, jump-diffusion, stochastic volatility), determine the symmetry group of the associated HJB and classify which are exactly solvable. This atlas would tell practitioners, once and for all, which stochastic control problems have closed-form solutions and which fundamentally do not.

The Master Gap

Each of the six gaps above is a paper. Gap 6 is the book.

The present work establishes that the boundary between *exactly solvable* and *numerically approximated* stochastic control problems has geometric structure — it is not random, not a matter of luck or inspiration, but a consequence of the symmetry group of the underlying HJB. KEGA found this boundary by reading the gap between Bluman and Pham. The next step is to map it completely.

This paper: one point on the boundary
Gap 6: the complete map of the boundary
The atlas: the book that does not yet exist

— Derived by applying KEGA to this paper's internal logic, 2026-03-31