

The Asymptotic Efficiency Theorem

A Mathematical Framework for Strategic Upgrade Valuation

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Abstract

In complex systems — from real-time strategy simulations to enterprise software architecture to capital investment — the decision to invest in a qualitative upgrade is frequently clouded by the perceived weight of the immediate cost. This paper introduces a rigorous cost-benefit framework to determine the exact **Break-Even Point** where a systemic upgrade becomes a mathematical necessity. We derive two complementary decision rules: a **resource threshold** model and a **maximum acceptable cost** model. By applying asymptotic analysis, we demonstrate that as total system resources grow without bound, any upgrade providing a positive efficiency gain inevitably renders its own fixed cost marginal, making the upgrade mandatory for optimal long-term output. The result is a general-purpose theorem applicable to any domain where fixed-cost investments yield proportional returns over a growing resource base.

I. Introduction

Consider a universal problem: you have a system that produces output. An upgrade is available that will improve the efficiency of every unit produced going forward, but it requires a one-time fixed cost. Should you invest?

This question arises everywhere:

- **Game theory:** Should a player research a technology upgrade when the cost diverts resources from immediate production?
- **Capital budgeting:** Should a firm invest in new machinery that improves throughput but requires upfront capital?
- **Software engineering:** Should a team refactor a codebase when the rewrite costs developer-hours but yields faster iteration forever after?
- **Education:** Should a student invest time learning a tool that accelerates all future work?

The intuition is straightforward — if the upgrade pays for itself, take it. But *when exactly* does it pay for itself? This paper formalizes the answer.

II. Definitions and Setup

Key Variables

Symbol	Name	Definition
C	Cost	The fixed, absolute resource units required to acquire the upgrade.
G	Gain Multiplier	The efficiency multiplier provided by the upgrade. A +20% improvement corresponds to $G = 1.20$.
E	Efficiency Gain	The percentage improvement, where $E = G - 1$. For $G = 1.20$, $E = 0.20$.
R	Remaining Resources	The total volume of units or value expected to benefit from the upgrade over the remaining lifetime of the system.

Output Functions

Without the upgrade, total effective output is simply:

$$O_{\text{without}} = R$$

With the upgrade, the system pays the fixed cost C, and the remaining resources (R - C) each benefit from the gain multiplier G:

$$O_{\text{with}} = (R - C) * G$$

The upgrade is economically justified when:

$$\begin{aligned} O_{\text{with}} &> O_{\text{without}} \\ (R - C) * G &> R \end{aligned}$$

This is the **Fundamental Inequality**.

III. The Break-Even Threshold (R-Model)

Derivation

Starting from the Fundamental Inequality:

$$(R - C) * G > R$$

Expanding:

$$R * G - C * G > R$$

Isolating R:

$$\begin{aligned} R * G - R &> C * G \\ R * (G - 1) &> C * G \end{aligned}$$

Solving for R:

$$R > (C * G) / (G - 1)$$

This is the **Break-Even Resource Threshold**. The upgrade becomes worthwhile when the remaining resources exceed this value.

Simplified Form

For small efficiency gains where $E = G - 1$, and when E is small relative to 1 (so G is approximately 1), the threshold simplifies to:

$$R > C / E$$

This is the intuitive form: **the number of future units multiplied by the efficiency gain per unit must exceed the cost of the upgrade.**

Worked Example: The Bloodlines Problem

In the real-time strategy game Age of Empires II, the technology "Bloodlines" provides a +20% hit point bonus to cavalry units. The research cost is equivalent to approximately 1.5 Knight units.

$$\begin{aligned} C &= 1.5 \text{ units} \\ E &= 0.20 \text{ (i.e., +20\%)} \\ G &= 1.20 \end{aligned}$$

Using the simplified form:

$$R > C / E = 1.5 / 0.20 = 7.5 \text{ units}$$

Decision rule: If the player expects to produce more than 7.5 cavalry units over the remainder of the game, researching Bloodlines is a net gain. If fewer than 7.5 units are expected, the cost outweighs the benefit.

Using the exact form:

$$R > (1.5 * 1.20) / (1.20 - 1.0) = 1.80 / 0.20 = 9.0 \text{ units}$$

The exact threshold is slightly higher because the simplified form neglects the opportunity cost of the resources consumed by the upgrade itself. For practical purposes with small E values, both forms converge.

IV. Maximum Acceptable Cost (C-Model)

The Inverse Question

Rather than asking "how many units justify the cost?", we can ask: **given a fixed total budget, what is the maximum percentage of resources that can be spent on an upgrade before it becomes non-viable?**

Derivation

Set the break-even condition where upgraded output equals base output:

$$(100 - C) * G = 100$$

Solving for C:

$$\begin{aligned} 100 - C &= 100 / G \\ C &= 100 - (100 / G) \end{aligned}$$

Or equivalently:

$$C = 100 * (1 - 1/G)$$

This gives C as a percentage of total resources.

Reference Table

Efficiency Gain (E)	Gain Multiplier (G)	Max Acceptable Cost (C %)
10%	1.10	9.09%
20%	1.20	16.67%
30%	1.30	23.08%
40%	1.40	28.57%
50%	1.50	33.33%
60%	1.60	37.50%
70%	1.70	41.18%
80%	1.80	44.44%
90%	1.90	47.37%
100%	2.00	50.00%

Interpretation: A +40% efficiency upgrade ($G = 1.40$) is only worth pursuing if the upgrade costs less than 28.57% of total available resources. Beyond that threshold, the cost consumes more value than the efficiency creates.

Notable boundary: Even a 100% efficiency gain (doubling output) can never justify spending more than 50% of total resources. The function $C = 100 * (1 - 1/G)$ is asymptotically bounded by 100% and approaches it only as G approaches infinity.

V. The Asymptotic Proof

Theorem Statement

The Asymptotic Efficiency Theorem: For any upgrade with a positive efficiency gain ($G > 1$) and a finite fixed cost (C), there exists a resource level R^* beyond which the upgrade is

always a net positive investment. Furthermore, as total resources grow without bound, the relative cost of the upgrade approaches zero, making the upgrade mandatory for optimal output in any expanding system.

Proof

We require:

$$R * (G - 1) > C * G$$

The right-hand side, $C * G$, is a fixed constant (since both C and G are constants of the upgrade).

The left-hand side, $R * (G - 1)$, grows linearly with R . Since $G > 1$, the coefficient $(G - 1)$ is strictly positive.

Therefore:

$$\lim (R \rightarrow \text{infinity}) [R * (G - 1)] = \text{infinity}$$

Since $\text{infinity} > C * G$ for any finite C and G , there exists a threshold:

$$R^* = (C * G) / (G - 1)$$

such that for all $R > R^*$, the inequality holds.

Moreover, the relative cost of the upgrade as a fraction of total resources is:

$$C / R$$

As R approaches infinity:

$$\lim (R \rightarrow \text{infinity}) [C / R] = 0$$

The fixed cost becomes infinitesimally small relative to the resource base. The gain, however, scales linearly with R. The net benefit diverges to infinity. **QED.**

Intuitive Summary

1. As R grows, the total gain ($R * E$) grows without bound.
2. The cost (C) remains constant.
3. The ratio of cost to gain collapses to zero.
4. Any positive efficiency gain, no matter how small, eventually overwhelms any finite fixed cost.

This is why long-lived systems — empires, companies, codebases, economies — should almost always invest in efficiency upgrades early. The earlier the upgrade, the larger the R that benefits from it.

VI. Extensions and Applications

6.1 Time-Discounted Resources

In real economic systems, future resources are discounted. If we apply a discount rate d , the effective remaining resources become:

$$R_{\text{effective}} = \text{sum over } t \text{ from } 1 \text{ to } T \text{ of } [r_t / (1 + d)^t]$$

The theorem still holds as long as $R_{\text{effective}}$ exceeds the break-even threshold. Higher discount rates raise the bar for justification but do not eliminate the asymptotic result for sufficiently long time horizons.

6.2 Stacking Multiple Upgrades

When multiple upgrades are available with multipliers $G_1, G_2, \dots, G_n$, the compound gain is:

$$G_{\text{total}} = G_1 * G_2 * \dots * G_n$$

The cost is the sum of individual costs. The break-even analysis applies to the compound system, and the asymptotic argument strengthens: compounding efficiency gains grow geometrically against linearly summed costs.

6.3 Diminishing Returns

If successive upgrades of the same type yield diminishing efficiency gains ($E_1 > E_2 > E_3 > \dots$), each subsequent upgrade has a higher break-even threshold. The theorem still guarantees each upgrade becomes worthwhile given sufficient scale, but the required scale increases with each iteration.

6.4 Broader Applications

Domain	Cost (C)	Gain (G)	Resources (R)
Software refactoring	Developer-hours for rewrite	Reduced bug rate, faster iteration	Total future feature-hours
Education	Time to learn a new tool	Productivity multiplier	Remaining career output
Manufacturing	Capex for new machinery	Units per hour improvement	Total future production volume
Infrastructure	Upfront build cost	Reduced per-unit transport cost	Total future throughput

In every case, the same inequality governs the decision, and the same asymptotic guarantee applies.

VII. Limitations

1. **Finite horizons:** The asymptotic guarantee assumes unbounded growth. In systems with hard resource caps or known termination points, the break-even threshold may never be reached.

2. **Opportunity cost of timing:** This model treats C as a lump sum. In practice, the timing of the investment matters — resources spent on an upgrade cannot simultaneously be deployed for immediate production. A full dynamic optimization would require modeling the production timeline.
 3. **Uncertainty:** The model assumes G and R are known. In reality, both the efficiency gain and the remaining resource base may be uncertain. Risk-adjusted versions of the theorem would incorporate expected values and variance.
 4. **Non-linear effects:** The model assumes linear scaling. Systems with saturation effects, bottlenecks, or nonlinear production functions may require modified formulations.
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VIII. Conclusion

The Asymptotic Efficiency Theorem provides a clean, general-purpose decision rule for upgrade investment:

1. **Compute the break-even threshold:** $R = (C * G) / (G - 1)$. If expected remaining resources exceed R , upgrade immediately.
2. **Compute the cost ceiling:** $C_{\text{max}} = 100 * (1 - 1/G)$. If the upgrade costs less than this percentage of total resources, it is viable.
3. **Trust the asymptote:** In any system with a long or growing horizon, positive-gain upgrades are not optional luxuries — they are mathematical necessities. The only question is timing.

The core insight is both intuitive and rigorous: **fixed costs are eaten alive by compounding gains over scale**. The earlier you upgrade, the more units benefit, and the larger your cumulative advantage. In the limit, refusing to upgrade when gain is positive is provably suboptimal.

References

1. The original framework was developed through analysis of upgrade economics in *Age of Empires II* (Ensemble Studios, 1999), where discrete technology investments yield percentage-based unit improvements.
2. The cost-benefit structure parallels classical capital budgeting theory. See: Brealey, R., Myers, S., & Allen, F. *Principles of Corporate Finance*, McGraw-Hill.
3. The asymptotic argument draws on standard limit analysis from real analysis. See: Rudin, W. *Principles of Mathematical Analysis*, McGraw-Hill.